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The Existence of Far Points in the Growth of the Sorgenfrey Line

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Abstract. In this paper, we will show that if X is a perfectly normal hereditary Lindeloff non-compact space with the following three properties: (1) The set of isolated points in X has a compact closure (2) The boundary of every open set in X is the derived set of a discrete subset contained in that open set and (3) X has an open base with cardinality less than or equal to c, then under the Continuum Hypothesis, each neighborhood of every point of $\beta X - X$ contains at least 2^c many far points of βX. As a corollary of this result, we conclude that $\beta\mathbb{R}_l - \mathbb{R}_l$ contains a dense set of exactly 2^c many far points.

Keywords: Tychonoff space; Remote point; Far point; Stone-Čech compactification.

1. Introduction

The remainder $\beta X - X$ of a Tychonoff space X has become a fascinating object of study since the Stone-Čech compactification βX was formally initiated. The construction of the Stone Čech compactification of Tychonoff space was condidered in [1]. One of the interesting problems in the study $\beta X - X$ is to investigate the existence of far points. Under the Continuum Hypothesis, N.J. Fine and L. Gillman [3] proved the existence of far points in $\beta\mathbb{R} - \mathbb{R}$. Fine and Gillman actually used the term "remote point" instead of "far point". We use here the terminology due to Eric Van Dowen [2].

Definition 1.1. *For a Tychonoff space X, a point $p \in \beta X$ is called a far point if it does not belong to the βX-closure of any discrete subset of X.*

In [6] D. Plank relates the the notion of far point and C-points for a class of spaces by the following two theorems. First we recall the following definition.

Definition 1.2. *For a Tychonoff space X, a point $p \in \beta X - X$ is said to be a C-point if for all $f \in C(X)$, $p \notin \partial(cl_{\beta X} Z(f) \cap (\beta X - X))$.*

Theorem 1.3. $[[6], 5.1]$ *Let X be a locally compact, real compact but non compact space. Then $intS^*(f) = (int_{\beta X} S(f)) \cap X^*$ for all $f \in C(X)$.*

Before stating the following theorem, let us first recall that in $C(X)$, $O^p = \{f \in C(X) : cl_{\beta X} Z(f)$ is a neighbourhood of p in $\beta X\}$.

Theorem 1.4. $[[6], 5.3]$ *Let $p \in \beta X - X$, where X is a metric space of nonmeasurable cardinal. Then the following conditions are equivalent:*
(1) *p is a $C-point$ of $\beta X - X$.*
(2) *$A^p = \{Z(f) \in Z[X] : f \in M^p\}$ has no member which is nowhere dense.*
(3) *$M^p = O^p$.*

If now p is a $far\ point$ of $\beta X - X$ then p is also a $C-point$ of $\beta X - X$. Conversely a $C-point$ of $\beta X - X$ is also a $far\ point$ $\beta X - X$ if we assume that the set of all isolated points in X has a compact closure in X.

Combining the above two theorems, Plank proved the following theorem.

Theorem 1.5. $[[6], 5.4]$ *If X is a separable, locally compact, non-compact metric space in which the set of isolated points has compact closure. Then βX contains a dense set of 2^c many far points which is a dense subspace of $\beta X - X$.*

In particular the above Theorem implies that $\beta\mathbb{R} - \mathbb{R}$ contains a dense subset of 2^c many far points. Being motivated by the work of Fine-Gillman [3] and Plank [6] and also using their mechanism of proofs, we will show in the following section that for the Sorgenfrey line $\mathbb{R}_l$, the growth $\beta\mathbb{R}_l - \mathbb{R}_l$ contains a dense subset of exactly 2^c many far point.

2. Existence of far Points in $\beta\mathbb{R}_l - \mathbb{R}_l$

In the proof of Theorem 5.3 of [6], we see that the hypothesis that X is a metric space is required only for two purposes (i) $Int_{\beta X}(cl_{\beta X}Z(f) \cap (\beta X - X)) = Int_{\beta X}(cl_{\beta X}Z(f)) \cap (\beta X - X)$ and (ii) every closed set is a zero set. Replacing the criteria that 'X is a metric space' by 'X is a perfectly normal realcompact space' we observe that the above two conditions hold again. This leads us to the following theorem.

Theorem 2.1. *Let X be a perfectly normal realcompact space and $p \in \beta X - X$. Then the following statements are equivalent.*

(1) *p is a $C-point$ of $\beta X - X$.*

(2) *$A^p = \{Z(f) \in Z[X] : f \in M^p\}$ has no member which is nowhere dense.*

(3) *$M^p = O^p$.*

If in addition, the boundary of every open set in X is the derived set of a discrete subset in the same open set then every far point of $\beta X - X$ is also a $C-point$ in the same, on the other hand if we assume that the set of all isolated points in X has a compact closure, then every $C-point$ of $\beta X - X$ is also a far point.

It is well known that in a metric space, the boundary of an open set can be represented as the set of all limit points of a suitable discrete subset contained in that open set. This result was proved by Hausdorff [5] long time ago and crucially used by Plank [6] to establish an important part of Theorem 1.4. In the following proposition, we will prove that Hausdorff's result is also valid in the Sorgenfrey line.

Proposition 2.2. *Let G be a non empty open set in $\mathbb{R}_l$. Then G has a decomposition: $G = \cup_n I_n$, where $\{I_n\}_n$ is either a finite or countably infinite family of mutually disjoint closed open intervals of the form $[a, b)$, $a < b$. Furthermore the boundary ∂G of G is the limit point of the set of the left end points of the I_n's. Finally the set of left end points of the I_n's makes a discrete subset of G.*

Proof. We choose $x \in G$ arbitrarily. Then there exists a closed open interval $[a_x, b)$ such that $x \in [a_x, b) \subset G$. Let I_x be the union of all closed open intervals containing x and contained in G. Then it is clear that I_x is open in $\mathbb{R}_l$ and is also an interval in the same (bounded or unbounded) with $G = \cup_x I_x$. Due to the maximality of I_x's, it is plain that for $x, y \in G$, either $I_x = I_y$ or $I_x \cap I_y = \varphi$. Since every subspace of $\mathbb{R}_l$ is Lindeloff, it follows that there are at most countably many disjoint I_x's in the above decomposition. Further, it is easy to prove that every interval can be expressed as $\cup_n J_n$ where J_n is of the form $[u, v)$ for some $u, v \in \mathbb{R}_l$ and $J_n \cap J_m = \varphi$. Therefore we can write $G = \cup_n I_n$ as a countable union of mutually disjoint closed open intervals of the form $[u, v)$. It is quite clear that each limit point of the set of the left end points of the intervals I_n's is a boundary point of G. On the other hand, let $x \in \mathbb{R}_l$ be a boundary point of G. Then x is obviously different from 'a_n' or 'b_n' for each $n \in \mathbb{N}$, where $I_n = [a_n, b_n)$. If possible assume x is not a limit point of the set of a_n, then there is a closed open interval $[x, x + \delta)$ which misses each 'a_n'. Now if there is an $n \in \mathbb{N}$ with $a_n < x < b_n < x+\delta$, then $[x, b_n)$ is an open neighbourhood of x in $\mathbb{R}_l$ disjoint with the complement of G. On the other hand if there does not exists any such n with either of these two properties then $[x, x+\delta)$ is disjoint with G. Thus in any case x can not belong to the boundary of G, a contradiction. Hence x is necessarily a limit point of the set of a_n's. The final portion of this Theorem is immediate because for each $n \in \mathbb{N}$, $[a_n, b_n)$ is an open neighbourhood of a_n in $\mathbb{R}_l$ missing each a_m with $m \neq n$. ∎

Since $\mathbb{R}_l$ is a perfectly normal realcompact space and is devoid of any isolated points, as a straightforward consequence of Theorem 2.1 and Proposition 2.2, we have the following theorem.

Theorem 2.3. *A point $p \in \beta\mathbb{R}_l - \mathbb{R}_l$ is a C-point of $\beta\mathbb{R}_l - \mathbb{R}_l$ if and only if it is a far point.*

Now we are going to show that there do exist *far points* in $\beta\mathbb{R}_l - \mathbb{R}_l$ in abundance. Our proof is in fact a straightforward generalization of Plank's theorem [[6].5.5]. We will require the following two results of Fine and Gillman [3].

Lemma 2.4. [[3], 2.1] *Let $f \geq 0$ be in $C(X) - C^*(X)$ and $(J_n)_{n\in\mathbb{N}}$ be a sequence of disjoint closed intervals in $\mathbb{R}$, each of which contains a value of f in its interior. Then the sets $E_n = \{x \in X : f(x) \in J_n\}$, $n \geq 1$ satisfy the following properties:*

(1) *Each E_n is a non-void zero set in X with non-void interior.*
(2) *If $(F_n)_n$ is a sequence of zero sets with $F_n \subset E_n$ for each $n \geq 1$, then $\cup_{n\in\mathbb{N}} F_n$ is a zero set.*

Lemma 2.5. [[3], 2.3][$Fine - Gillman$] *Suppose that X is a non-pseudocompact space containing not more than $\aleph_1$ dense open sets with $\{U_\alpha : \omega_0 \leq \alpha \leq \omega_1\}$ as the family of these dense open sets. Then for any disjoint sequence of zero sets $\{E_n : n \in \mathbb{N}\}$ in X, each with non-void interior, it is possible to construct a family $\{Z_\alpha : \alpha < \omega_1\}$ of non-void zero sets in X which possess the finite intersection property such that $Z_\alpha \subset U_\alpha$ for each $\omega_0 \leq \alpha \leq \omega_1$ and $Z_\alpha \subset \cup_{n\in\mathbb{N}} E_n$ for each $0 \leq \alpha < \omega_1$. From the above, it follows that there exists a free z-ultrafilter on X no member of which is nowhere dense.*

Theorem 2.6. *Let X be a perfectly normal hereditary Lindelöf non-compact space with the following three properties:*

(1) *The set of isolated points in X has a compact closure.*
(2) *The boundary of every open set in X is the derived set of a discrete subset contained in that open set.*
(3) *X has an open base with cardinality less than or equal to c.*

Then under the Continuum Hypothesis, each neighbourhood of every point of $\beta X - X$ contains at least 2^c many far points of βX.

Proof. X being Lindelöf it is realcompact. We choose a point $q \in \beta X - X$ and a zero set neighbourhood V of q in the space βX. To exhibit a far point of βX in V it is sufficient to produce a free z-ultrafilter A^p on X, no member of which is nowhere dense such that $p \in V$. Since X is realcompact there exists an $h \in C(X)$, $h \geq 0$ such that $h^*(q) = \infty$, where h^* is the Stone extension [3,7.5 and 8.19]. Now choose $g \in C^*(X)$, $g \geq 0$ such that $g^\beta(q) = 1$ and $g^\beta(\beta X - V) = \{0\}$, g^β being the Stone extension of g (Recall that since g is bounded, $g^* = g^\beta$). Set $f = g \cdot h$. Then it is clear that $f^*(q) = \infty$ and f is unbounded on $V \cap X$. Also $V \cap X$ is a zero set of X with nonempty interior. Hence using Lemma 2.4, we can choose a disjoint sequence $(E_n)_n$ of zero sets in X each contained $V \cap X$ satisfying the following two properties:

(1) Each E_n has non-void interior in X.

(2) If for each $n \in \mathbb{N}$, $F_n \in Z(X)$ is such that $F_n \subset E_n$ then $\cup_{n\in\mathbb{N}} F_n$ is a zero set X.

For each $n, k \in \mathbb{N}$, set $E_n^k = E_{2^{k-1}(2n-1)}$. Then each sequence $(E_n^k)_n$ also satisfies the conditions of Lemma 2.4. Since we have assumed the Continuum Hypothesis, we can take the entire family of dense open sets in X as $\{U_\alpha : \omega_0 \leq \alpha < \omega_1\}$. Therefore for each $k \in \mathbb{N}$, we can apply Lemma 2.5 to obtain a family $\{Z_\alpha^k : 0 \leq \alpha < \omega_1\}$ of zero sets in X with finite intersection property such that $\cap_{\alpha<\omega_1} Z_\alpha^k = \emptyset$ which further satisfies the properties: (i) $Z_\alpha^k \subset U_\alpha$ for each $\omega_0 \leq \alpha < \omega_1$ and (ii) $Z_\alpha^k \subset \cup_{n\in\mathbb{N}} E_n^k$ for each $0 \leq \alpha < \omega_1$. For $0 \leq \alpha < \omega_1$, we note that $Z_\alpha = \cup_{k\in\mathbb{N}} Z_\alpha^k$. Then $Z_\alpha \subset U_\alpha$ for $\omega_0 \leq \alpha < \omega_1$ and for each $0 \leq \alpha < \omega_1$, Z_α is a zero set in X. Therefore the family $\Im = \{Z_\alpha : 0 \leq \alpha < \omega_1\}$ of zero sets in X can be extended to a free z-ultrafilter A^p on X for some point $p \in \beta X - X$. Since each dense open set in X contains some member of the family $\Im$, it is clear that no member of A^p is nowhere dense. Hence from Theorem 2.1, it follows that p is a far point of βX. We now observe that for each $k \in \mathbb{N}$, $\Im \cup \{Z_0^k\}$ has the finite intersection property. As $(Z_0^k)_{k\in\mathbb{N}}$ is a disjoint family of zero sets in X, it therefore follows that $\Im$ can be extended to infinitely many points $p \in \beta X - X$ for which $\Im \subset A^p$ and of course each such point p is a far point of βX. Now for each Z in $\Im$, $Z \subset V \cap X$, by Gelfand-Kolmogoroff theorem, $p \in cl_{\beta X} Z$ implies that $p \in cl_{\beta X} V = V$. Let $S = \{p \in \beta X - X : \Im \subset A^p\}$. Then again by the Gelfand-Kolmogoroff theorem, $S = \cap_{Z\in\Im} cl_{\beta X} Z$; which is an infinite compact subset of $\beta X - X$, in particular S is a non discrete closed subset of $\beta X - X$ and so contains a copy of $\beta\mathbb{N}$. Therefore $|S| \geq 2^c$. Thus the Theorem is completely proved. ∎

Remark 2.7. Since the Sorgenfrey line satisfies all the three conditions of the above Theorem, it follows under the Continuum Hypothesis that $\beta\mathbb{R}_l - \mathbb{R}_l$ contains a dense subset of at least 2^c many far points. But since $\mathbb{R}_l$ is a separable space and every separable compact Hausdorff space is a continuous image of $\beta\mathbb{N}$, it follows that $|\beta\mathbb{R}_l| \leq |\beta\mathbb{N}| = 2^c$. Hence $|\beta\mathbb{R}_l| = 2^c$ and so $\beta\mathbb{R}_l - \mathbb{R}_l$ contains a dense set of 2^c many far points of $\beta\mathbb{R}_l$.

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